Car sequencing is NP-hard: a short proof

Abstract

In this note, a new proof is given that the car sequencing problem is NP-hard. Established from the hamiltonian path problem, the reduction is direct while closing gaps remaining in the previous NP-hardness results. Since car sequencing is studied in many operational research courses, this result and its proof are particularly interesting for teaching purposes.
Keywords: Car sequencing, Computational complexity, NP-hardness proof

Introduction

The car sequencing problem consists in scheduling cars along an assembly line composed of different posts where are installed the equipments and options relative to each vehicle (radio, sun-roof, air-conditioning, etc). In order to smooth the workload at these posts, the cars for which setting options needs some heavy operations are spaced out in the sequence. In other words, the goal is to minimize the density of cars in the sequence which require much work to assemble. This need of spacing out cars is formalized by defining a ratio constraint for each option. For example, for an option with ratio 3/7, one shall not find more than 3 cars affected by the option in any subsequence consisting of 7 consecutive cars. Then, the objective is to find a sequence of cars satisfying all ratio constraints.

Given its economic issues, car sequencing was heavily studied these last twenty years. In 2005, the French Operations Research Society, jointly with the car manufacturer Renault, posed a real-life car sequencing problem as subject of an international OR competition, leading to a large body of research. Estellon *et al* (2006, 2008) have provided state-of-the-art practical solution approaches to car sequencing in this context. The reader is referred to Solnon *et al* (2008) for a survey on the topic.

In its simplest form, the car sequencing (CS) problem can be formally defined as follows. An instance is composed of n cars and m options. Each option o_k represents a ratio constraint p_k/q_k with $p_k < q_k \leq n$. Each car c_i is defined as a bit string of length m such that $c_{i,k} = 1$ if car i has option k, and 0 otherwise. A solution is given as a sequence (that is, a permutation) of the n cars such that for each option o_k, every subsequence of q_k cars contains at most p_k cars requiring o_k. Another way to define the car sequencing problem (CS') is to consider in input the cars grouped according to the different configurations induced by the options. In this case, the input is composed of $n_1, \ldots, n_k$ cars from k configurations, instead of the arbitrary n cars.

Gent (1998) proved that CS is NP-complete, by reduction from the Hamiltonian Path (HP) problem. However, his reduction is rather complex and his paper remains unpublished. Kis (2004) established that CS' (and thus CS) is strongly NP-hard, from the Exact Cover by 3-Sets problem. In addition, Kis observed that the compactness of CS' instances has surprising consequences in terms of computational complexity. First, contrarily to CS, car sequences do not necessarily induce polynomial certificates in CS'. Then, he devised a pseudo-polynomial algorithm (by dynamic programming) for solving CS' when the number of configurations is fixed and denominators of ratio constraints are not too large.

In this note, a refined NP-hardness result is provided through a straightforward parsimonious reduction from the Hamiltonian Path problem. All terms related to computational complexity which are not defined here can be found in Garey and Johnson (1979) and Welsh and Gale (2001).

The complexity result

Kis (2004) shows that car sequencing is NP-hard, even if all options have ratios with denominator at most 4. Indeed, his reduction uses options with ratios 1/4 or 2/3 only. Here the following result is established directly.

Theorem 1. *CS is NP-complete, even if all options have ratios 1/2.*

Proof. The reduction is performed from Hamiltonian Path, well-known to be NP-complete (Garey and Johnson, 1979). An instance of HP is composed of an undirected graph $G(V, E)$ with n vertices. A solution is given as a path in G visiting each vertex exactly once. Now, an instance of CS is built as follows. For each vertex i, a car c_i is defined. For each pair (i, j) of non adjacent vertices, an option $o_{i,j}$ with ratio 1/2 is defined and assigned to cars c_i and c_j. Clearly, such a transformation is done in polynomial time. Then, two cars do not share an option, and thus can be sequenced consecutively, if and only if their corresponding vertices in the graph are adjacent. Consequently, a one-to-one mapping exists between all hamiltonian paths and all admissible car sequences. $\square$

Remark 1. In this reduction, the size of the CS instance is not linear in the size of the HP instance, since the number of options is quadratic in n for sparse graphs. It is possible to obtain only n options by slightly modifying the construction as follows. For each vertex i, a car c_i is defined having the option o_i with ratio 1/2. Now, for each pair (i, j) of non adjacent vertices, the car c_i has option o_j and the car c_j has option o_i. In this case, the cars/options matrix corresponds exactly to the one's complement of the adjacency matrix of G.

Remark 2. These two reductions can be easily adapted to show directly that CS' is NP-hard, even if all options have ratios 1/2 (each option induces actually one configuration). Since HP is #P-hard (Welsh and Gale, 2001, p. 118), the one-to-one mapping between hamiltonian paths and admissible car sequences implies that CS and CS' are #P-hard too.

As conclusion, we mention the following issue which remains open: what is the complexity of the car sequencing problem when the number of options (or of configurations) is fixed and some denominators of ratio constraints are large?

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