4OR manuscript No.
(will be inserted by the editor)

Mutual exclusion scheduling with interval graphs or related classes: complexity and algorithms

Frédéric Gardi[⋆]

PROLOGIA S.A. - Groupe Air Liquide, Parc Scientifique et Technologique de Luminy, case 919, bâtiment CCIMP, F-13288 Marseille cedex 9, France.
e-mail: `frederic.gardi@prologia.fr`

Received: — / Revised version: —

Abstract This note summarizes the main results presented in the author's Ph.D. thesis, supervised by Professor Michel Van Caneghem and defended on 14th June 2005 at University of Aix-Marseille II, France. The thesis, written in French, is available at `http://www.lif-sud.univ-mrs.fr/Rapports/25-2005.html`. The mutual exclusion scheduling problem has an elegant graph-theoretic formulation: given an undirected graph G and an integer k, find a minimum coloring of G such that each color appears at most k times. When G is an interval graph, this problem has some applications in workforce planning. Then, the object of the thesis is to study the complexity of mutual exclusion scheduling problem for interval graphs and related classes.

Keywords: mutual exclusion scheduling, graph coloring, workforce planning, interval graphs, graph classes.

MSC classification: 05C15, 90B35, 68W40, 68R10, 68Q25.

---

1 Introduction

The *mutual exclusion scheduling problem* can be formulated as follows in graph-theoretic terms: given an undirected graph G and an integer k, find a minimum coloring of G such that each color appears at most k times. The problem owns this name because of its important applications in the

⋆ On leave from Laboratoire d'Informatique Fondamentale - CNRS UMR 6166, Parc Scientifique et Technologique de Luminy, Marseille, France.

scheduling domain. When G is an *interval graph*, this one has some applications in *workforce planning* too. In effect, we have been confronted to it during the development of the software BAMBOO for ressources planning, edited by the firm PROLOGIA of the Groupe Air Liquide (for more details on BAMBOO, see `http://www.prologia.fr/bamboo/bamboo.html`).

The problem encountered is the following. Consider a set of daily tasks, each one having a starting date and an ending date. An employee is able to execute correctly a set of tasks if they do not overlap during the day. For several reasons (regulation of work, security, maintenance of machines), an employee must not execute more than k tasks in a day (generally $k \leq 5$). Then, the question is: how employees have to be mobilized to complete all the tasks? Obviously, a planning describing which tasks have to be assigned to each employee is required. Since each task is only an interval of time, the problem amounts to color the underlying interval graph such that each color appears no more than k times, which corresponds exactly to the *mutual exclusion scheduling problem for interval graphs.* When the planning is cyclic (the same tasks recur each day and some of them overlap two consecutive days), we obtain the same problem but for *circular-arc graphs.* When there are not enough employees to execute all the tasks (e.g., because some employees are absent), it is interesting to allow overlapping between certain tasks during the assignment. In this case, the problem relates to *tolerance graphs.*

$\mathcal{NP}$-hard in the general case, the complexity of the mutual exclusion scheduling problem has been investigated for several classes of graphs, in particular some classes of perfect graphs (Baker and Coffman, 1996; Bodlaender and Fomin, 2004; Bodlaender and Jansen, 1995; Cohen and Tarsi, 1991; Dahlhaus and Karpinski, 1998; De Werra, 1997; Finke et al., 2004; Hansen et al., 1993; Jansen, 2003; Jarvis and Zhou, 2001; Lonc, 1991). Unfortunately, few positive results have been published on the subject: for the majority of the classes studied, the problem is shown to be $\mathcal{NP}$-hard. Bodlaender and Jansen (1995) have notably established that the problem restricted to interval graphs remains $\mathcal{NP}$-hard even if k is a constant greater or equal than 4. We took up the problem at this point and the object of this thesis is to study in details the complexity of mutual exclusion scheduling problem for interval graphs and for some related classes, like circular-arc graphs or tolerance graphs.

Our contribution stands at two levels. We exhibit several polynomial cases significant in practice, for which we have been careful to devise some simple and efficient algorithms (in particular linear-time and space algorithms). On the other hand, by reinforcing the existing $\mathcal{NP}$-hardness results, we obtain a precise cartography of the complexity of the problem for the classes of graphs studied.

2 New results on complexity and algorithms

First, the complexity of the mutual exclusion scheduling problem for interval graphs is detailed. A new algorithm, much simpler than the one of Andrews et al. (2000), is proposed to solve in linear time and space the problem when $k = 2$. At the same time, some related problems are investigated like maximum matching in convex bipartite graphs or minimum coloring of interval graphs, for which new optimal-time and space algorithms are given. Finally, we show that the problem is linear-time and space solvable for two well-known subclasses of interval graphs using greedy-like algorithms: proper interval graphs and threshold graphs.

Then, the problem is investigated for the two extensions of interval graphs which are circular-arc graphs and tolerance graphs. An $O(n^2)$-time and linear-space algorithm is proposed to solve the problem restricted to proper circular-arc graphs, as well as a linear-time and space algorithm for the same problem when $k = 2$. Then, the Bodlaender and Jansen's theorem is reinforced by establishing that when $k \geq 3$ the problem remains $\mathcal{NP}$-hard for bounded tolerance graphs, even if the stable set of largest cardinality in the graph has $k + 1$ vertices and that any cycle of length greater or equal than five has two chords. A corollary of this result is that the problem remains $\mathcal{NP}$-hard for Meyniel graphs and weakly triangulated graphs whose complement is also a partially ordered graph, even for fixed $k \geq 3$.

3 A sufficient condition for optimality

Motivated by practical aspects, we study the impact of the following property on the complexity of the problem: the graph G admits a coloring such that each color appears at least k times. In a workforce planning context, this hypothesis is often verified. This is notably the case when working schedules of municipal bus drivers or airport personnels are planned: the rotations of bus or planes generally induce some packets of consecutive tasks of size greater or equal than k (for reasonable values like $k \leq 5$). We show that if an interval graph owns this property, then it can be optimally partitioned into $\lceil n/k \rceil$ stable sets of size at most k. Moreover, if a coloring of the interval graph which satisfies the property is given as input, then the mutual exclusion scheduling problem is solved in linear time and space. Surprisingly, this assertion is extended to circular-arc graphs, claw-free graphs, proper tolerance graphs (for $k = 2$) and chordal graphs (for $k \leq 4$). As a consequence of the result concerning claw-free graphs, we obtain that the problem is polynomial for perfect claw-free graphs, and in particular for line-graphs of weakly triangulated graphs.

4 On partitioning problems related to interval graphs

The last chapter of the thesis is devoted to some partitioning problems related to interval graphs, inspired from the study of the mutual exclusion scheduling problem. We particularly linger over the problem of partitioning interval graphs into proper interval (induced) subgraphs, for which the following proposition is established: any interval graph admits a partition into less than $\lceil \log_3 n \rceil$ proper interval subgraphs and this partition is computed in $O(n \log n + m)$ time and linear space. This result is used to devise a polynomial approximation algorithm for the mutual exclusion scheduling problem restricted to interval or circular-arc graphs. We propose two algorithms reaching in linear time and space an approximation ratio lower than 2 in real-life situations (i.e., under certain conditions). Finally, three other problems are investigated: partitioning an interval graph into the least number of cliques or stable sets, determining the largest proper interval subgraph of an interval graph, finding the minimal number of different lengths of intervals necessary to represent an interval graph.

References

Andrews MG, Atallah MJ, Chen DZ, Lee DT (2000). Parallel algorithms for maximum matching in complements of interval graphs and related problems. *Algorithmica* 26: 263–289.

Baker BS, Coffman EG Jr (1996). Mutual exclusion scheduling. *Theoretical Computer Science* 162:225–243.

Bodlaender HL, Fomin FV (2004). Equitable colorings of bounded treewidth graphs. In: Fiala J, Koubek V, Kratochvíl J (eds) *Proceedings of MFCS 2003, the 29th International Symposium on Mathematical Foundations of Computer Science. Lecture Notes in Computer Science* 3153:180–190. Springer-Verlag, Berlin.

Bodlaender HL, Jansen K (1995). Restrictions of graph partition problems. Part I. *Theoretical Computer Science* 148:93–109.

Cohen E, Tarsi M (1991). $\mathcal{NP}$-completeness of graph decomposition problem. *Journal of Complexity* 7:200–212.

Dahlhaus E, Karpinski M (1998). Matching and multidimensional matching in chordal and strongly chordal graphs. *Discrete Applied Mathematics* 84:79–91.

De Werra D (1997). Restricted coloring models for timetabling. *Discrete Mathematics* 165/166:161–170.

Finke G, Jost V, Queyranne M, Sebö A (2004). Batch processing with interval graph compatibilities between tasks. Cahier 108. Laboratoire Leibniz (IMAG), Grenoble, France. `http//www-leibniz.imag.fr/LesCahiers/`

Hansen H, Hertz A, Kuplinski J (1993). Bounded vertex colorings of graphs. *Discrete Mathematics* 111:305–312.

Jansen K (2003). The mutual exclusion scheduling problem for permutation and comparability graphs. *Information and Computation* 180(2):71–81.

Jarvis J, Zhou B (2001). Bounded vertex coloring of trees. *Discrete Mathematics* 232: 145–151.

Lonc Z (1991). On complexity of some chain and antichain partition problems. In Schmidt G, Berghammer R (eds) *Proceedings of WG'91, the 17th International Workshop on Graph–Theoretic Concepts in Computer Science. Lecture Notes in Computer Science* 570:97–104. Springer-Verlag, Berlin.