A note on the Roberts characterization of proper and unit interval graphs

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Abstract

In this note, a constructive proof is given that the classes of proper interval graphs and unit interval graphs coincide, a result originally established by Fred S. Roberts. Additionally, the proof yields a linear-time and space algorithm to compute a unit interval representation, given a proper interval graph as input.

Key words: proper interval graph, unit interval graph, constructive proof, efficient representation

1 Introduction

An undirected graph G with vertex set $V(G)$ and edge set $E(G)$ is an *interval graph* if to each vertex $v \in V(G)$ a closed interval $I_v = [l_v, r_v]$ of the real line can be associated, such that two distinct vertices $u, v \in V(G)$ are adjacent if and only if $I_u \cap I_v \neq \emptyset$. The family $\{I_v\}_{v \in V(G)}$ is an *interval representation* of G. An undirected graph G is a *proper interval graph* if there is an interval representation of G in which no interval properly contains another. In the same way, an undirected graph G is a *unit interval graph* if there is an interval representation of G in which all the intervals have the same length. For more details about the world of interval graphs, the reader can consult [6,7].

In 1969, Roberts [14] proved that the classes of proper interval graphs and unit interval graphs coincide. He showed notably that $K_{1,3}$-free interval graphs are unit interval graphs by using the Scott-Suppes characterization of semiorders [15]. Then, the trivial implications "unit $\Rightarrow$ proper $\Rightarrow$ $K_{1,3}$-free" for interval

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graphs enabled him to establish the whole result. Recently, Bogart and West [1] gave a constructive proof of this result, where proper intervals are gradually converted into unit intervals by means of successive contractions, dilations and translations.

In this note, a new constructive proof of the Roberts characterization of proper and unit interval graphs is presented. As in the Bogart-West proof [1], the unit interval representation is produced from left to right by processing a representation of the graph. Here the proof is completely combinatorial, without need for coordinate manipulation: the unit interval model is built directly from the clique-vertex incidence matrix, which can be easily obtained from the proper interval representation. The correctness of the construction relies crucially on the fact that the clique-vertex incidence matrix of a proper interval graph has the consecutive 1s property both for rows and for columns. We recall that a $(0,1)$-matrix has the consecutive 1s property for columns (resp. rows) if its rows (resp. columns) can be permuted in such a way that the 1s in each column (resp. row) occur consecutively. As a conclusion, some computational issues are discussed.

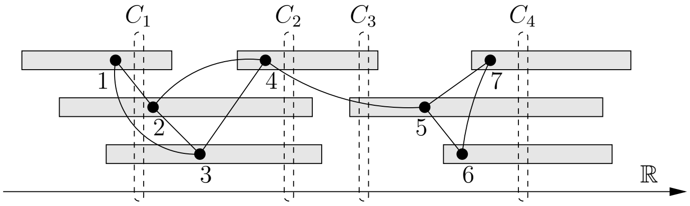
For other characterizations of proper and unit interval graphs, the reader is referred to the seminal works of Wegner [16] and Roberts [13]. All graph-theoretical terms not defined here can be found in [7]. The numbers of vertices and edges of the graph G are denoted respectively by n and m throughout the paper.

2 The proof

Theorem 1 *For an undirected graph G, the following statements are equivalent:*

(1) G is a proper interval graph,
(2) the clique-vertex incidence matrix of G has the consecutive 1s property both for rows and for columns,
(3) G is a unit interval graph,
(4) G is a $K_{1,3}$-free interval graph.

Proof. (1) $\Rightarrow$ (2). When no interval properly contains another, the left-endpoint order and the right-endpoint order are the same. Let u and v be the first and the last vertices of a maximal clique under this ordering. The interval I_u extends far enough to the right to reach the left endpoint of I_v, and the same is true of every assigned interval that starts between the left endpoints of I_u and I_v. Hence the clique consists precisely of these consecutive vertices in the ordering (see Figure 1). Since the cliques occupy intervals of consecutive vertices in this ordering, and maximality of the cliques implies that no such interval contains another, putting the cliques in increasing order of their first vertices in the vertex order also establishes consecutivity of the set of cliques containing any vertex.



	1	2	3	4	5	6	7
C_1	1	1	1	0	0	0	0
C_2	0	1	1	1	0	0	0
C_3	0	0	0	1	1	0	0
C_4	0	0	0	0	1	1	1
	a_1	a_2		a_3	a_4		
			b_1	b_2	b_3		b_4

Fig. 1. A proper interval graph and its clique-vertex incidence matrix.

(2) $\Rightarrow$ (3). From the consecutive 1s property for given orderings of the rows and columns of the clique-vertex incidence matrix, the i^{th} clique C_i consists of consecutive vertices $v_{a(i)}, \ldots, v_{b(i)}$ from the vertex ordering, with each of $\langle a \rangle$ and $\langle b \rangle$ being a strictly increasing sequence (see Figure 1). Initially, represent the vertices of C_1 by $b(1)$ pairwise intersecting distinct unit intervals whose left endpoints are in the same order as the indexing $v_1, \ldots, v_{b(1)}$. For $j > 1$, having assigned unit intervals to the vertices of $\bigcup_{i<j} C_i$ to represent the subgraph they induce, we add distinct unit intervals for $C_j - C_{j-1}$ to extend this to a unit interval representation of the subgraph induced by $\bigcup_{i \le j} C_i$.

We specify $I_v = [l_v, r_v]$ for $v \in C_j - C_{j-1}$ by putting $l_{b(j-1)+1}, \ldots, l_{b(j)}$, in order, between $r_{a(j)-1}$ and $r_{a(j)}$. Since $v_{a(j)-1} \in C_{j-1}$, we have $l_{b(j-1)} < r_{a(j)-1}$, and therefore the left endpoints occur in the desired order. Extend the resulting intervals to unit length, setting $r_k = l_k + 1$ for $b(j-1) + 1 \le k \le b(j)$. All these intervals end after $r_{b(j-1)}$. Since the members of $C_j - C_{j-1}$ are adjacent to all of $v_{a(j)}, \ldots, v_{b(j-1)}$ and to no earlier vertices, we have a unit interval representation of the desired subgraph.

Since the implications (3) $\Rightarrow$ (4) $\Rightarrow$ (1) are well known and easy to obtain (see [1] for example), the proof is complete. $\square$

Note. Item (2) of Theorem 1 was previously stated in different language in [6, p. 85] and can be also formulated as follows: there exists a linear ordering $v_1, \ldots, v_n$ of $V(G)$ such that for all $i < j$, $v_i v_j \in E(G)$ implies that all the vertices between v_i and v_j in this ordering induce a clique. It is worth noting that this last assertion is the natural strengthening for proper interval graphs of an earlier characterization of interval graphs that appears in all of [8,10,12]: the existence of an ordering $v_1, \ldots, v_n$ such that for all $i < j$, $v_i v_j \in E(G)$ implies that $v_k v_j \in E(G)$ whenever $i < k < j$.

3 Computational issues

Given a proper interval graph, many applications require knowing a unit interval representation of the graph. Generally, linear-time and space recognition algorithms for proper interval graphs produce only the linear ordering described above for $V(G)$ (see [2–5,9,11] for example). Here we discuss how to compute a unit interval model efficiently, given a proper interval graph G and the linear ordering on $V(G)$.

Having the linear ordering on the vertices, the ordered set of maximal cliques is easily computed in linear time and space. Then, the construction given in the proof of the implication (2) $\Rightarrow$ (3) of Theorem 1 yields a linear-time and space algorithm to compute a unit interval representation. This is more efficient than the Bogart-West construction [1], which computes a unit interval model in $O(n^2)$ time from a proper interval representation.

However, these representations are not efficient in the sense that the endpoints of the unit intervals are some arbitrary rationals which may have denominators exponential in n. Only Corneil et al. [3] have shown that their breadth-first search recognition algorithm could be used to construct a unit interval model in which each endpoint is rational, with denominator n and numerator lower than n^2. Thus, it would be interesting to find a linear-time and space algorithm which computes directly (that is, without the use of breadth-first search) an efficient unit interval model, given a proper interval graph G and the linear ordering on $V(G)$ as input.

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